Factor-based Dynamic Models and its application to macroeconomic forecasting

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Introduction

- In the recent macroeconomic literature factor-based dynamic model have become popular.
- We can mention two recent important working papers where this approach has been successfully applied.

1. Stock-Watson (2002) performed forecasting experiments for USA macroeconomic variables using 215 explanatory variables.
2. Forni et al. (2000, 2003) provided a time-series forecasting method based on spectral analysis, an then applied this approach to forecast Euro-area main macroeconomic indicators using 447 explanatory variables.

All these papers find that the mean squared errors of estimates and forecasts based on factor models are lower than those obtained from univariate and multivariate AR models.

The structure of the presentation

- The main idea of the factor-based dynamic model
- Estimation factor dynamics using static principal component analysis method
- Estimation factor dynamics using dynamic principal component analysis method
- Two models for forecasting macroeconomic variables
- Data description
- Forecast comparisons
- Conclusions

Factor-based dynamic model

- The whole process of creating factor-based dynamic model can be divided on the two steps:
- The factor estimation step and the forecasting step.
- In factor models variables are represented as the sum of two mutually orthogonal unobservable components: the common components and the idiosyncratic component.
- Let X_t be the (Nx1) dimensional vector of stationary time series with observations for t = 1,2,…, T, and it is assumed that the series have zero mean and covariance Γ_0. The factor model representation is given by:

$$X_t = \varkappa(F_t) + \xi_t$$

The idea behind the factor model is that a small number r of factors F_t should be able to explain most of the variance of the data $r << N$.

In practice two methods for determining factors are commonly used, in particular Stock-Watson (SW or Static principal component approach) and Forni , Hallin Lippi and Reichlin (FHLR or dynamic principal component approach) approach. Let's present this approaches in turn.

Static principal component analysis (Stoch-Watson approach)

- The factor model proposed by Stock and Watson uses static principal component analysis to derive the factors.
- According to static principal component analysis approach the factors can be calculated in the following manner.
- Let’s assume that we have X_{1t}, X_{2t}, …, X_{nt} initial set of time series, where t = 1, 2, …, T. We assume that there exist a linear combinations among these initial set of variables, which can be presented F = XB, where F is the vector of unobserved factors, X is the vector of initial variables and B is the unknown matrix of parameters that we need to estimate.
- The B matrix of parameters should satisfy the following two conditions: The first condition is that var(X) = var(F) or var(X) = var(XB) → var(X) = B’SB, where S is the variance-covariance matrix of the initial vector of variables. The second important condition is that the B matrix of parameters should be derived in a such way, in order to $var(F_1) > var(F_2) > … > var(F_n)$.
- As it was mentioned the number of extracted factors should be less than the number of initial variables, but the extracted factors should explain the most part of variances of the initial vector variables.

Static principal component analysis (Stoch-Watson approach)

- Imposing restriction on the matrix B , that is B'B = 1 and solving maximization problem we are able to find the following matrix equation.

$$R = B'SB - \lambda(B'B - 1)$$

$$dR/dB = d(B'SB) - \lambda(d(B'B)) = (dB)'SB + B'S(dB) -$$
$$\lambda((dB)'B + B'(dB)) = 2B'S(dB) - 2\lambda B'(dB) = 0$$

$$B'S - \lambda B' = 0 \rightarrow SB - \lambda B = 0$$

$$|S - \lambda I|B = 0$$

  Thus we need to solve $|S - \lambda I| = 0$, because $B \neq 0$.

- To solve $|S - \lambda I| = 0$ matrix equation means to find the Eigen values λ and corresponding eigenvectors B. Thus according to Stock and Watson approach factors can be determined as F = BX, where B is the eigenvectors of the variance-covariance matrix of the initial vector of variables.

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

- The factor model proposed by Forni, Hallin, Lippi and Reichlin (2000, 2003) uses dynamic principal component analysis to derive the factors.
- Similar to the Stock-Watson factor model it is assumed that the vector of the initial time series $X_t = (X_{1t}, X_{2t}, \ldots, X_{nt})'$ is the sum of the two mutually orthogonal unobservable component: the common component $\varkappa_t$ and the idiosyncratic component ξ_t.
- Similar to the Stock-Watson factor model we need to find out how much of X_t variance is explained by the variance of common components and how much by the variance of idiosyncratic components.

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

- The estimation of common and idiosyncratic components is conducted in two steps.
- In the first step we need to calculate the spectral density matrix for different frequencies. Using calculated spectral density matrices we compute the covariance matrices for the common and idiosyncratic components.
- In the second step using calculated covariance matrices for common and idiosyncratic components we can construct a linear combination for the current period using the generalized principal component algorithm.
- Now let’s present the above mentioned two steps in more details.

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

First step:

1. Using initial vector of time series $X_t = (X_{1t}, X_{2t}, \ldots, X_{nt})'$ we need to compute autocovariance matrices of order k.

$$\Gamma_{nk}^{T} = \left(\frac{1}{T-k}\right)\sum_{t=k+1}^{T} x_{n,t}x_{n,t-k}^{'}$$

   Where Γ_{nk}^{T} is the covariance matrices of order k, n is the number of time series used in the computation, and T is the number of periods in the data set. As a result we obtain matrices within the range $\Gamma_{n(-k)}^{T}, \ldots, \Gamma_{n(0)}^{T}, \ldots, \Gamma_{n(k)}^{T}$.

2. After that the spectral density matrix is computed

$$\Sigma_{n}^{T}(\theta_s) = \frac{1}{2\pi}\sum_{k=-M}^{M} w_k \Gamma_{nk}^{T} e^{-i\theta_s k} \quad (1)$$

   Where $\Sigma_{n}^{T}(\theta_s)$ is (n,n) matrices for the spectrum $\theta_s = \frac{2s\pi}{2M+1}$ where s = -M, -M+1, …, M, M = round(sqrt(M)) and $w_k = 1 - \frac{|k|}{M+1}$

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

3. In this step we need to compute the spectral decomposition matrices using eigenvalue and eigenvector extraction methodology. Thus we can derive $\sum^{\chi}(\theta_s)$ and $\sum^{\xi}(\theta_s)$

$$\begin{aligned}\textstyle\sum^{\chi}(\theta) &= \lambda_1(\theta)p_1(\theta)p_1(\theta)'+....+\lambda_q(\theta)p_q(\theta)p_q(\theta)' \\ \textstyle\sum^{\xi}(\theta) &= \lambda_{q+1}(\theta)p_{q+1}(\theta)p_{q+1}(\theta)'+....+\lambda_n(\theta)p_n(\theta)p_n(\theta)'\end{aligned} \quad (2)$$

From $\sum^{\chi}(\theta_s)$ and $\sum^{\xi}(\theta_s)$ matrices, using the inverse Fourier transformation, we can derive covariance matrices for common and idiosyncratic components.

4. Covariance matrices for common and idiosyncratic components can be calculated as

$$\Gamma_{nk}^{\chi T} = \frac{2\pi}{2M+1}\sum_{h=-M}^{M}\Sigma_n^{\chi T}(\theta_s)e^{i\theta_h k} \quad \Gamma_{nk}^{\xi T} = \frac{2\pi}{2M+1}\sum_{h=-M}^{M}\Sigma_n^{\xi T}(\theta_s)e^{i\theta_h k} \quad (3)$$

Where, $\Gamma_{nk}^{\chi T}$ is covariance matrices for common component, $\Gamma_{nk}^{\xi T}$ is covariance matrices for idiosyncratic component

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

It is can be shown that $\Gamma_{n0}^{\chi T} + \Gamma_{n0}^{\xi T} = \Gamma_{n0}^{T}$. When k = 0, then for any θ_s the spectral density matrix is equal $\sum_n^T(\theta_s) = \frac{1}{2\pi}\Gamma_{n0}^T$ Therefore we can write the following identity.

$$\Gamma_{n0}^{\chi T} + \Gamma_{n0}^{\xi T} = \frac{2\pi}{2M+1}\left(\sum_{h=-M}^{M}\Sigma_n^{\chi T}(\theta_s) + \sum_{h=-M}^{M}\Sigma_n^{\xi T}(\theta_s)\right) =$$

$$\frac{2\pi}{2M+1}((2M+1)\Sigma_n^T(\theta_s)) = \frac{2\pi}{2M+1}\frac{2M+1}{2\pi}\Gamma_{n0}^T = \Gamma_{n0}^T$$

But when k ≠ 0, then $\Gamma_{n0}^{\chi T} + \Gamma_{n0}^{\xi T} \neq \Gamma_{n0}^{T}$. In order to show let's suppose that k = 1, then we will have.

$$\Gamma_{n1}^{\chi T} + \Gamma_{n1}^{\xi T} = \frac{2\pi}{2M+1}\left(\sum_{s=-M}^{M}\frac{\omega_1\Gamma_{n1}^T e^{-i\theta_s}}{2\pi}\right) = \frac{1}{2M+1}\sum_{s=-M}^{M}\omega_1\Gamma_{n1}^T e^{-i\theta_s}$$

Dynamic Principal component analysis (Forni, Hallin, Lippi and Reichlin approach)

Thus using the inverse Fourier transformation it is possible to decompose initial covariance matrices on the two sub matrices, particularly common component covariance matrix and the idiosyncratic component covariance matrix.

Second Step:

1. Using $\Gamma_{n0}^{\chi T}$ and $\Gamma_{n0}^{\xi T}$ it is possible to calculate generalized Eigen values and eigenvectors $Z_n^T = (Z_{n1}^{T'}, \dots, Z_{nr}^{T'})$
2. The generalized eigenvalues and eigenvectors can be computed from the following matrix equation. This matrix can be obtained the similar to Stoc and Watson optimization procedure.

$$\left| \Gamma_{n0}^{\chi T} (\Gamma_{n0}^{\xi T})^{-1} - \lambda_{nj}^{T} \mathrm{I}_{T \times T} \right| (Z_{nj}^{T})' = 0$$

3. The obtained variables allow for computing matrix

$$\hat{\chi}_{t+k}^{gdfm} = \Gamma_{nk}^{\chi T} \hat{Z}' (\hat{Z} \Gamma_{n0}^{T} \hat{Z}')^{-1} \hat{Z} x_t$$

Using this equation it is possible to conduct forecasting for the k periods ahead. As we can see the dynamic principal component analysis comparing with the static is relatively difficult.

Comparing this two factor extraction algorithms we are able to decide which algorithm is more suitable for the forecasting purposes. Therefore in order to answer on question which method gives more accurate forecast results we need to compare this two approaches.

Two forecasting models

- In the paper by Stock and Watson (2002) the forecasting equation has the following general form.

$$\hat{y}^{h}_{T+h|T} = \hat{\alpha}_h + \sum_{j=1}^{m} \hat{\beta}_{hj} \hat{F}_{T-j+1} + \sum_{j=1}^{p} \hat{\gamma}_{hj} y_{T-j+1}$$

- Where F is the vector of k common factors estimated from the whole panel of candidate predictor series X_t using static principal component analysis.

  In the paper by Forni et al. (2000) the forecasting equation has the following form.

$$\hat{y}^{h}_{T+h|T} = \hat{\alpha}_h + \sum_{j=1}^{m} \hat{\beta}_{hj} \hat{F}_{T-j+1}$$

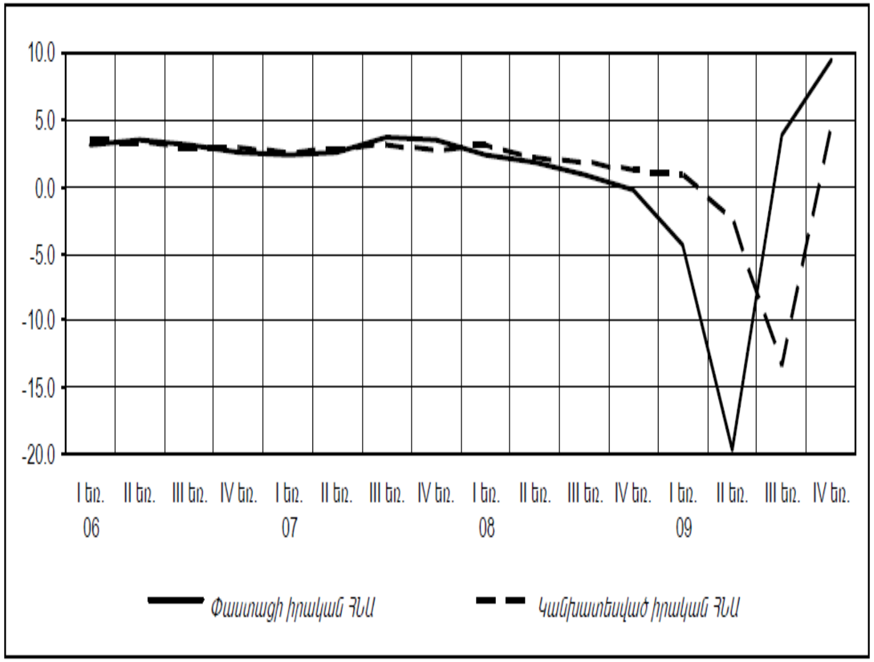
  Where F is the vector of common factors estimated by dynamic principal component analysis.

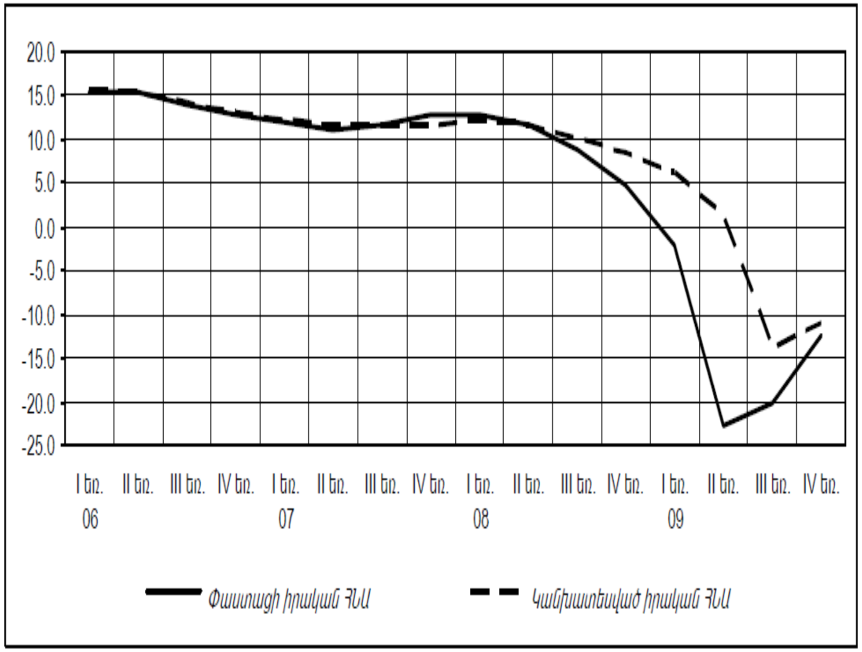
  As we can see in the model by Forni et al. we have not lagged values of the dependent variables. Thus our purpose is to describe which forecasting models and which factor extraction algorithms could give more accurate forecast results.

Data description

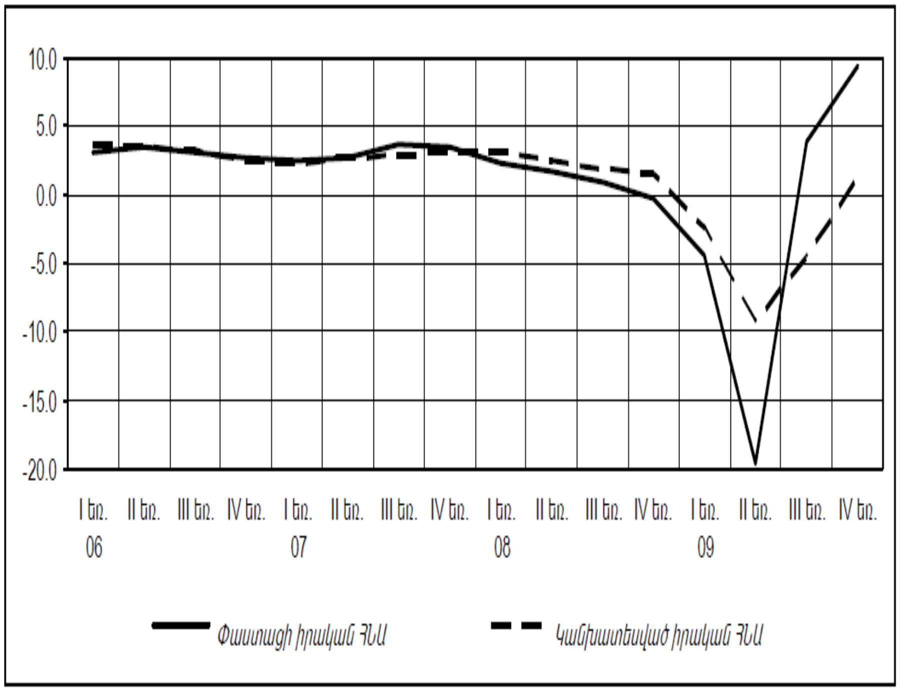
- Our data consists of quarterly time series of 40 macroeconomic time series from 2000/Q2 – 2009/Q4.
- For forecasting real GDP growth rate and inflation rate have been used 38 actual macroeconomic time series, particularly in the data set included variables on national accounts (12), consumer prices and exchange rate (11), financial and monetary policy variables (11) and international indicators (4).
- All initial time series have been preliminary treated, that is we take log of the difference and some of them seasonally adjust (TRAMO/SEATS seasonal adjustment procedure).
- All calculation relating with static and dynamic principal component analysis, as well as estimation and forecasting have been done with using specially created codes in the space of MATLAB.

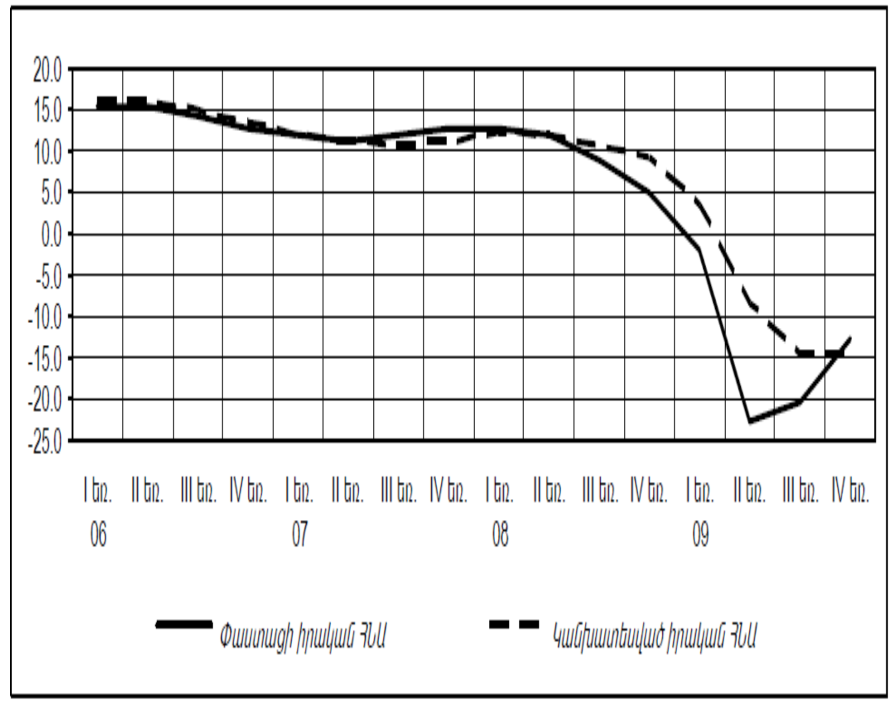
Forecasting results: Actual and Forecasted values for the real GDP growth rate 2006-2009 (in %-th with respect to the previous quarter, and in % with respect to the same quarter of the previous year) Stock-Watson (SW) factor model



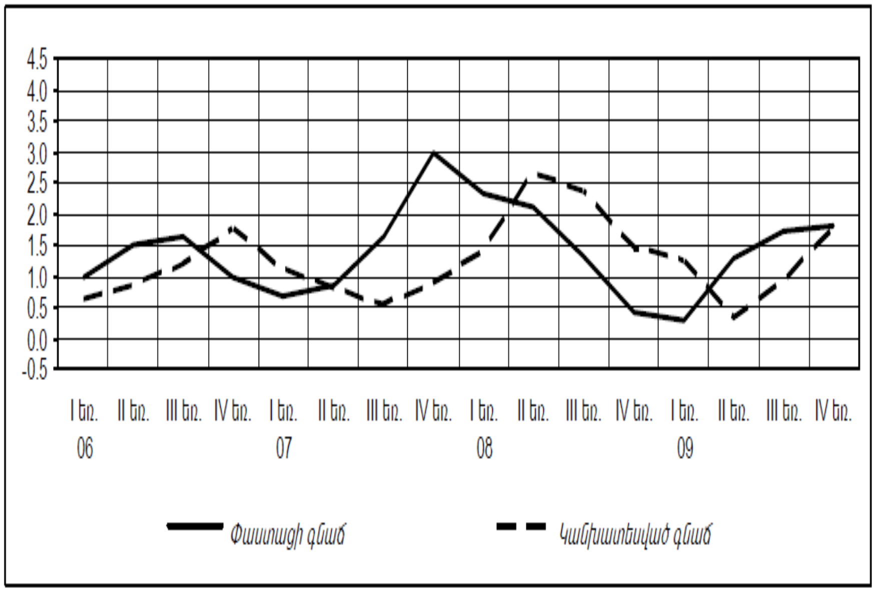


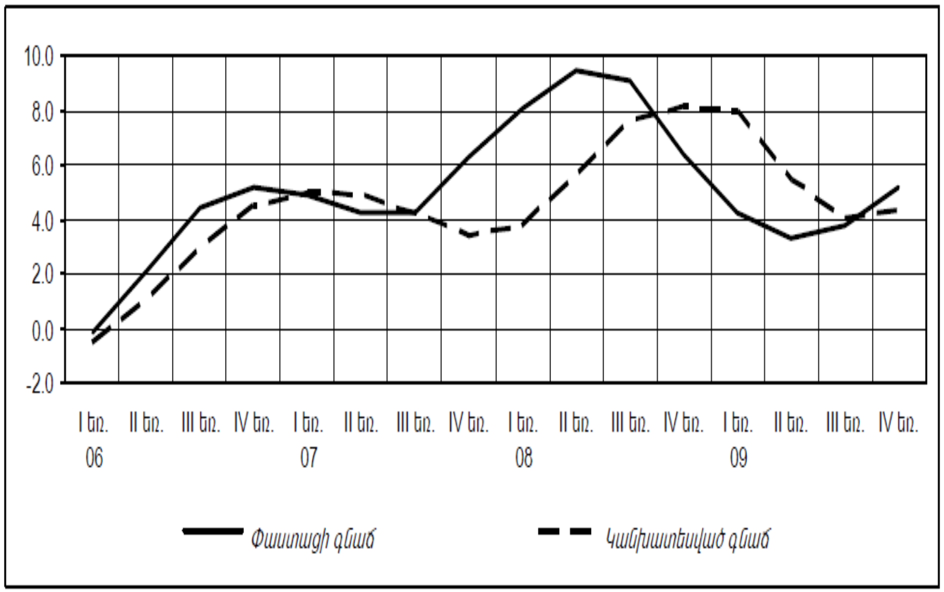
Forecasting results: Actual and Forecasted values for the real GDP growth rate 2006-2009 (in %-th with respect to the previous quarter, and in %-th with respect to the same quarter of the previous year) Forni et al. (FHLR) factor model



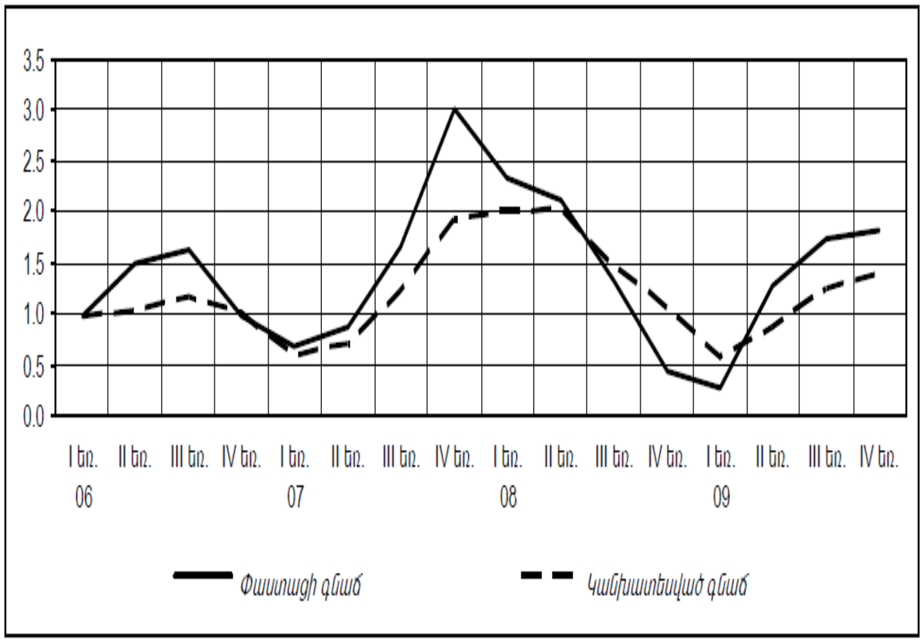


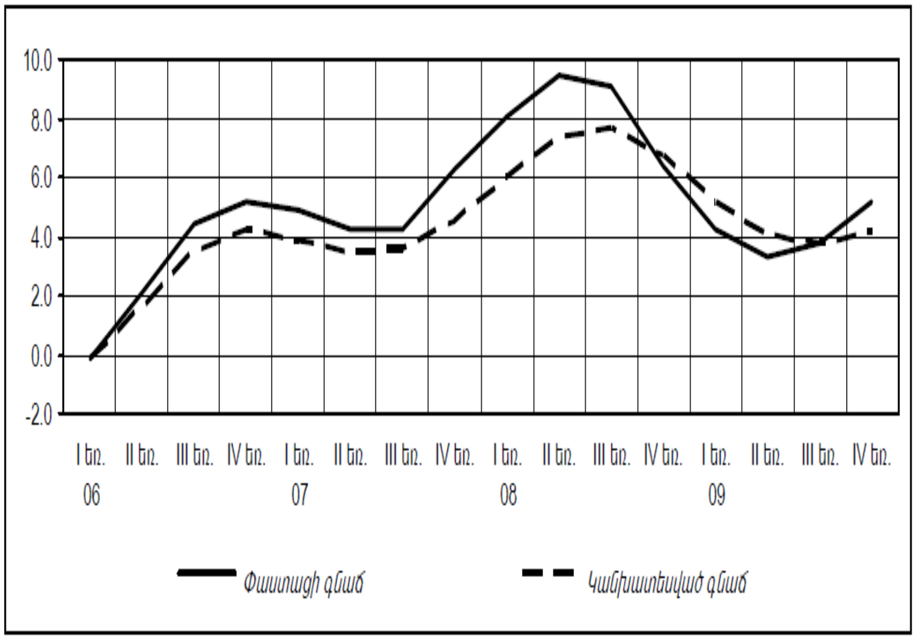
Forecasting results: Actual and Forecasted values for the inflation rate 2006-2009 (in %-th with respect to the previous quarter, and in %-th with respect to the same quarter of the previous year) Stock-Watson (SW) factor model





Forecasting results: Actual and Forecasted values for the inflation rate 2006-2009 (in %-th with respect to the previous quarter and in %-th with respect to the same quarter of the previous year) Forni et al. (FHLR) factor model





Conclusions

- From the above presented figures we can see that the FHLR forecasting model gives more accurate forecast results, while the SW model results are shifted for one lag ahead. This is depends from the chosen models for forecasting.
- If we select for both factor estimation methods the same forecasting model then we can see that SW model gives more accurate forecasting results than the FHLR model.
- Both model can be used for short-term forecasting purposes and the resulting forecast can be obtained as a averaged value of the separate forecasts obtained by SW and FHLR factor models.